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Jauch–Piron Logics with Finiteness Conditions

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We show that there are no non-Boolean block-finite orthomodular posets possessing a unital set of Jauch–Piron states. Thus, an orthomodular poset representing a quantum physical system must have infinitely many blocks.

1. INTRODUCTION AND PRELIMINARIES

An event structure (so-called "quantum logic") of a quantum mechanical system is commonly assumed to be an orthomodular poset L. A state of such a system is then interpreted as a probability measure on L. It turns out that the orthomodular posets which may potentially serve as "logics" must have reasonably rich spaces of states. Moreover, the following condition on the state space appears among the axioms of a quantum system: if Φ is a state on a logic L, and $\Phi(a)=\Phi(b)=1$ for some $a, b \in L$, then there is a $c \in L$ such that $c \leq a$, $c \leq b$, and $\Phi(c)=1$. Such a state is said to be a Jauch–Piron state. If all states on L fulfil this condition, then L is called a Jauch–Piron logic. The condition was originally introduced by Jauch (1968) and Piron (1976).

We investigate unital Jauch–Piron logics with finitely many blocks (maximal Boolean subalgebras). We show that such a logic is always Boolean, i.e., it represents a purely classical system. In other words, an orthomodular poset must have infinitely many blocks in order to describe a (nonclassical) quantum system.

This generalizes the result of Rüttimann (1977) concerning finite Jauch–Piron orthomodular lattices, and the result of Bunce *et al.* (1985) for finite Jauch–Piron logics (not necessarily lattices). On the other hand, there is a non-Boolean unital Jauch–Piron logic whose blocks are finite (moreover, uniformly bounded)—consider the set of all projections on a Hilbert space

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of dimension three. Thus, the theorem of Rüttimann cannot be weakened in this direction.

Let us start by reviewing basis notions and facts [see, e.g., Kalmbach (1984) and Pták and Pulmannová (1989) for details].

Definition 1.1. A *logic* is a partially ordered set L with a least and a greatest element 0, 1 together with an operation $'$ (an orthocomplementation) such that the following conditions are satisfied for any $a, b \in L$ (the symbols $\vee$, $\wedge$ mean the lattice operations induced by $\leq$):

(i) $(a')' = a$.
(ii) $a \leq b$ implies $b' \leq a'$.
(iii) $a \vee a' = 1$.
(iv) If $a \leq b'$, then $a \vee b$ exists in L.
(v) If $a \leq b$, then $b = a \vee (a' \wedge b)$ (the orthomodular law).

Definition 1.2. Let a and b be in a logic L. They are said to be *orthogonal* (in symbols, $a \perp b$) if $a \leq b'$, and they are said to be *compatible* (in symbols, $a \leftrightarrow b$) if $a = c \vee e$ and $b = d \vee e$, where $c, d, e \in L$ are mutually orthogonal. An element $a \in L$ is called *central* if a is compatible with every $b \in L$. The set of all central elements of L is called the *center* of L and will be denoted by $C(L)$. If $C(L) = \{0, 1\}$, we say that L has a trivial center. Let $a, b \in L$, $a \leq b$. Then the *interval* $[a, b]_L$ in L is defined as $[a, b]_L = \{x \in L \mid a \leq x \leq b\}$. (The subscript is omitted if this does not cause any misunderstanding.)

It is a well-known fact (Pták and Pulmannová, 1989) that a logic L is a Boolean algebra if and only if every pair of its elements is compatible.

Definition 1.3. A *block* of a logic L is a maximal Boolean subalgebra of L. A logic L is said to be *block-finite* if the system of all blocks of L is finite.

Block-finite logics were thoroughly studied by Bruns and Greechie (1982*a*,*b*).

Definition 1.4. Let K, L be logics and let f: $K \to L$ be a mapping. Then f is called a *logic morphism* if the following conditions hold true:

(i) $f(0) = 0$.
(ii) $f(a') = f(a)'$ for any $a \in K$.
(iii) $f(a \vee b) = f(a) \vee f(b)$ whenever $a, b \in K$ are orthogonal.

Definition 1.5. A *state* on a logic L is a mapping Φ: $L \to \langle 0, 1 \rangle$ such that:

(i) $\Phi(1) = 1$.
(ii) If $a, b \in L$ and $a \perp b$, then $\Phi(a \vee b) = \Phi(a) + \Phi(b)$.

Let us denote by $\mathcal{S}(L)$ the set of all states on L (called the "state space").

The set $\mathcal{S}(L)$ is naturally endowed with a topological and convex structure (as a subset of $\langle 0, 1 \rangle^L$). In fact, we have the following result.

Proposition 1.6 (see Shultz, 1974). State spaces are (up to affine homeomorphisms) exactly compact convex subsets in locally convex topological linear spaces.

We shall need the following simple property of state spaces described first by Godowski (1982).

Proposition 1.7. Let K, L be logics and $f: K \to L$ a logic morphism.

(i) If $\Psi \in \mathcal{S}(L)$, then $\Psi \circ f \in \mathcal{S}(K)$.

(ii) Suppose further that $f(a) \leftrightarrow f(b)$ implies $a \leftrightarrow b$. If $\Phi \in \mathcal{S}(K)$ such that $\Phi(x) = 0$ whenever $f(x) = 0$, then the mapping $\Psi: L \to \langle 0, 1\rangle$ defined by $\Psi(f(a)) = \Phi(a)$ is a state on L.

Proof. (i) Let $\Psi \in \mathcal{S}(L)$ and put $\Phi = \Psi \circ f$. Now $\Phi(1) = \Psi(f(1)) = \Psi(1) = 1$. If $a, b \in K$, $a \perp b$, then $f(a) \perp f(b)$ and

$$\begin{aligned}\Phi(a \vee b) &= \Psi(f(a \vee b)) = \Psi(f(a) \vee f(b)) \\ &= \Psi(f(a)) + \Psi(f(b)) = \Phi(a) + \Phi(b)\end{aligned}$$

Hence $\Phi \in \mathcal{S}(K)$. (ii) First we must prove that Ψ is properly defined. Let $a, b \in K$ such that $f(a) = f(b)$. Then $a \leftrightarrow b$ and there are mutually orthogonal elements $c, d, e \in K$ such that $a = c \vee e$, $b = d \vee e$. Now $f(c), f(d), f(e)$ are mutually orthogonal elements in L and $f(c) \vee f(e) = f(a) = f(b) = f(d) \vee f(e)$. Hence $f(c) = 0 = f(d)$, $\Phi(c) = 0 = \Phi(d)$, and $\Phi(a) = \Phi(e) = \Phi(b)$. Now we can show that Ψ is a state on L. We have $\Psi(1) = \Psi(f(1)) = \Phi(1) = 1$. Let $f(a), f(b) \in L$ such that $f(a) \perp f(b)$. Then $a \leftrightarrow b$, $f(a \wedge b) = 0$, $f(a \wedge b') = f(a)$, and $f(a' \wedge b) = f(b)$. Since $(a \wedge b') \perp (a' \wedge b)$, we get

$$\begin{aligned}\Psi(f(a) \vee f(b)) &= \Psi(f(a \wedge b') \vee f(a' \wedge b)) = \Psi(f((a \wedge b') \vee (a' \wedge b))) \\ &= \Phi((a \wedge b') \vee (a' \wedge b)) = \Phi(a \wedge b') + \Phi(a' \wedge b) \\ &= \Psi(f(a \wedge b')) + \Psi(f(a' \wedge b)) = \Psi(f(a)) + \Psi(f(b)) \quad \blacksquare\end{aligned}$$

2. BLOCK-FINITE LOGICS

In this section we introduce our notation for block-finite logics and prove some helpful lemmas.

Let L be a block-finite logic. Let $\mathcal{B} = \{B_1, B_2, \ldots, B_n\}$ be the set of all blocks of L. Denote $\mathcal{C}(L) = \{C = A_1 \cap A_2 \cap \cdots \cap A_n \mid A_i = B_i \text{ or } A_i = L - B_i$ for $i = 1, 2, \ldots, n\}$. Then $C(L) \in \mathcal{C}(L)$ and each $x \in L$ is an element of just one set of $\mathcal{C}(L)$. For each $C \in \mathcal{C}(L)$ put

$$D_C = \{x \in C \mid (y \in L, y \neq 0, y \leq x) \Rightarrow y \in C\}$$

We call a (finite or infinite) sequence $a_1 \geq a_2 \geq a_3 \geq \cdots$ in L *down-changing* if $a_i \in C$ $[C \in \mathcal{C}(L)]$ implies $a_{i+1} \notin C$ for each $i = 1, 2, 3, \ldots$. For every $b \in L$ we denote $E_b = \{g \in L \mid g \leq b$ and $g \in \bigcup_{C \in \mathcal{C}(L)} D_C\}$. We call the elements of E_b *bottom elements* (for b).

Lemma 2.1. Let L be a block-finite logic, $a, b, x \in L$, $a, b \in C$ for some $C \in \mathscr{C}(L)$. Then $x \leftrightarrow a$ if and only if $x \leftrightarrow b$.

The proof of Lemma 2.1 is straightforward.

Lemma 2.2. The set $D_{C(L)}$ forms an ideal, i.e.:

(i) If $x \in D_{C(L)}$, $y \in L$, $y \leq x$, then $y \in D_{C(L)}$.

(ii) If $x, y \in D_{C(L)}$, then $x \vee y \in D_{C(L)}$.

Proof. Part (i) is guaranteed by the definition. (ii) If $x, y \in D_{C(L)}$, then $x \vee y$ exists and belongs to $C(L)$. Let $z \in L$, $z \leq x \vee y$. We put $z_1 = z \wedge x$, $z_2 = z \wedge y \wedge x'$. Now $z_1, z_2 \in D_{C(L)}$ and thus $z = z_1 \vee z_2$ is a central element. ■

Lemma 2.3. Let L be a block-finite logic. Then each down-changing sequence in L is finite.

Proof. Suppose, in contradiction, that there is an infinite down-changing sequence $a_1 \geq a_2 \geq a_3 \geq \cdots$ in L. Then there are $C_\alpha, C_\beta \in \mathscr{C}(L)$ such that each of them contains an infinite subsequence of $a_1 \geq a_2 \geq a_3 \geq \cdots$. Take a subsequence $b_1 \geq b_2 \geq b_3 \geq \cdots$ of $a_1 \geq a_2 \geq a_3 \geq \cdots$ such that $b_{2k-1} \in C_\alpha$, $b_{2k} \in C_\beta$ for $k = 1, 2, 3, \ldots$. There are blocks B_α, B_β in L such that $B_\alpha \cap C_\alpha = \emptyset$, $B_\beta \cap C_\alpha = C_\alpha$, $B_\alpha \cap C_\beta = C_\beta = B_\beta \cap C_\beta$ (if necessary, we interchange the roles of C_α and C_β). Now Navara and Rogalewicz (1991), Proposition 5.7 implies that the following blocks exist in L:

$$B_2 = [0, b_2]_{B_\alpha} \times [0, b_2']_{B_\beta},$$
$$B_4 = [0, b_4]_{B_\alpha} \times [0, b_4']_{B_\beta},$$
$$B_6 = [0, b_6]_{B_\alpha} \times [0, b_6']_{B_\beta}, \quad \text{etc.}$$

We have $b_{2i-1} \notin B_{2k}$ for $k \geq i$ and $b_{2i-1} \in B_{2k}$ for $k < i$. Thus, we have constructed an infinite sequence of different blocks in L, which is in a contradiction with its block-finiteness. ■

Corollary 2.4. Let L be a block-finite logic. If $b \in L$, $b \neq 0$, then there is a bottom element $c \in E_b$, $c \neq 0$.

3. JAUCH–PIRON LOGICS

Definition 3.1. A state Φ on a logic L is said to be a *Jauch–Piron state* if the following implication is satisfied: if $\Phi(a) = 1 = \Phi(b)$ for $a, b \in L$, then there is $c \in L$ such that $c \leqq a$, $c \leqq b$, and $\Phi(c) = 1$.

Definition 3.2. (i) A logic L is called a *Jauch–Piron logic* if every state on L is a Jauch–Piron state.

(ii) A logic L is said to be *unital* if, for each $a \in L$, $a \neq 0$, there is a $\Phi \in \mathscr{S}(L)$ with $\Phi(a) = 1$.

The following three theorems belong to the main results of Bunce *et al.* (1985), where the proofs can be found. Let us only notice that Theorem

3.3 is an easy consequence of Proposition 1.7, and the proof of Theorem 3.4 follows immediately from the description of states on a product of logics (Maňasová and Pták, 1981). The proof of Theorem 3.5 is quite nontrivial. The geometry of the state space (its convex structure) is widely utilized. The theorem was first proved by Rüttimann (1977) for logics which are lattices, and then generalized for all finite logics (and a little simplified) in Bunce *et al.* (1985).

Theorem 3.3. Let f be a logic morphism of K onto L, where K is a Jauch–Piron logic. Then L is also a Jauch–Piron logic.

Theorem 3.4. Let $\{L_\alpha \mid \alpha \in I\}$ be a family of (unital) logics. Let I be a set whose cardinal is not real-measurable. Then $L = \prod_{\alpha \in I} L_\alpha$ is a (unital) Jauch–Piron logic if and only if L_α is a (unital) Jauch–Piron logic for every $\alpha \in I$.

Theorem 3.5. Let L be a finite unital Jauch–Piron logic. Then L is a Boolean algebra.

Now we are ready to prove our main result. We start with a lemma.

Lemma 3.6. Let L be a block-finite unital Jauch–Piron logic, $a \in L$. Then all bottom elements for a are central.

Proof. We divide the proof into two steps. First, we prove that the assumptions of the theorem imply that card $D_C \leq 1$ for all $C \in \mathscr{E}(L)$, $C \neq C(L)$. Second, we shall prove that $D_C = \varnothing$ if $C \neq C(L)$. Before starting with Step 1, let us notice a tiny observation: If $C_1, C_2 \in \mathscr{E}(L)$, $C_1 \neq C_2$, and $c \leftrightarrow d$ for some $c \in D_{C_1}$, $d \in D_{C_2}$, then $c \perp d$. This follows from the fact that for $e = c \wedge d$ we have $e \leq c$, $e \leq d$, and thus either $e = 0$ or $e \in D_{C_1} \cap D_{C_2}$. Since $D_{C_1} \cap D_{C_2} = \varnothing$, we get $c \perp d$.

Step 1. Suppose that there is $C_a \in \mathscr{E}(L)$, $C_a \neq C(L)$, and $a, b \in D_{C_a}$, $a \neq b$. We can assume $a \perp b$. (Since $a, b \in C_a$, we have $a \leftrightarrow b$. Let us write $a = c \vee e$, $b = d \vee e$ for $c, d, e \in L$, $c \perp d$, $c \perp e$, $d \perp e$. Now $c \leq a$, $d \leq b$, $e \leq b$, and $a, b \in D_{C_a}$, which implies that $c, d, e \in D_{C_a}$. At least two of them are different from zero and we take them for a, b.) Then there exists $d \in L$, $d \not\leftrightarrow a$, $d \in D_{C_d}$ for some $C_d \in \mathscr{E}(L)$.

The proof of this seems to require the compactness of the state space (Proposition 1.6). Since $C_a \neq C(L)$, there exists $d \in L$, $d \not\leftrightarrow a$. We denote C_d the class in $\mathscr{E}(L)$ containing d. Due to Lemma 2.3, we can assume that d is chosen such that $u \leftrightarrow a$ for every $u \leq d$, $u \notin C_d$. Suppose that d is not a bottom element. For each $C \in \mathscr{E}(L)$ we denote $A_C = E_d \cap C$. We shall show that for each $C \in \mathscr{E}(L)$ there is $\Psi \in \mathscr{S}(L)$ such that $\Psi(d) = 1$, $\Psi(a) = 0$, and $\Psi(u) = 0$ whenever $u \in A_C$. For every $u \in A_C$ we denote $\mathscr{S}_u(L) = \{\Phi \in \mathscr{S}(L) \mid \Phi(u) = 0,\ \Phi(a) = 0,\ \Phi(d) = 1\}$. Since $u \leq d$, $u \neq d$, there exists

$v \in E_d$, $v \perp u$, and because $v \notin C_a$ and $v \leftrightarrow a$, also $v \perp a$. If $\Phi(v) = 1$, then $\Phi \in \mathcal{S}_u(L)$. Thus, $\mathcal{S}_u(L)$ is a nonvoid closed subset of the compact space $\mathcal{S}(L)$. If $u, v \in A_C$, then $u \vee v \in A_C$ and $\mathcal{S}_{u \vee v}(L) \subseteq \mathcal{S}_u(L) \cap \mathcal{S}_v(L)$. Hence $\bigcap_{u \in A_C} \mathcal{S}_u(L) \neq \emptyset$. This proves the existence of Ψ.

We construct such a state for each $C \in \mathscr{C}(L)$ with $A_C \neq \emptyset$, and denote them $\Psi_1, \Psi_2, \ldots, \Psi_p$. Now let $\Psi = (1/p) \sum_{i=1}^{p} \Psi_i$. We have $\Psi(a') = 1$ and $\Psi(d) = 1$. If $u \leq d$, $u \notin C_d$, then there is $v \in E_d$, $v \leq d \wedge u'$. We have $\Psi_i(v) = 0$ for some $i \in \{1, 2, \ldots, p\}$, and therefore $\Psi(v) \leq (p-1)/p$. By the Jauch–Piron property, there exists $e \in L$ such that $e \leq d$, $e \leq a'$, and $\Psi(e) = 1$. Since $a' \not\leftrightarrow d$, it follows that $e \notin C_d$. This is a contradiction with the former result. Thus, d is a bottom element, i.e., $d \in D_{C_d}$.

Recall that we have $a, b \in D_{C_a}$, $a \perp b$, and $d \in D_{C_d}$, $d \not\leftrightarrow a$. Let $\Phi \in \mathcal{S}(L)$ such that $\Phi(d) = 1$. Denote $\Phi(a) = A$, $\Phi(b) = B$. We have $A, B \notin \{0, 1\}$. [If $\Phi(a) = 1$, then, from the Jauch–Piron property, there is $x \in L$ such that $x \leq a$, $x \leq d$, and $\Phi(x) = 1$. On the other hand, $a \in D_{C_a}$, $d \in D_{C_d}$, $C_a \neq C_d$, and hence the only element under a and d is 0—a contradiction. A similar argument can be repeated for a', b, and b'.] Define a mapping Ψ: $L \to \langle 0, 1 \rangle$ as follows:

(i) If $x \not\leftrightarrow a$, then $\Psi(x) = \Phi(x)$.

(ii) If $x \leftrightarrow a$, then $x = x_a \vee x_b \vee x_1$ for $x_a \leq a$, $x_b \leq b$, $x_1 \leq (a \vee b)'$, and we put

$$\Psi(x) = \Phi(x_1) + \frac{A+B}{A} \cdot \Phi(x_a)$$

We claim that Ψ is a state on L. Since $\Psi(1) = 1$, we must prove that $\Psi(e \vee g) = \Psi(e) + \Psi(g)$ for any $e, g \in L$, $e \perp g$. If e, g, and $e \vee g$ are all compatible with a, or all noncompatible with a, then this equality is straightforward. Suppose $e \leftrightarrow a$, $g \not\leftrightarrow a$. Then either $e \perp a \vee b$ and $\Psi(e \vee g) = \Phi(e \vee g) = \Phi(e) + \Phi(g) = \Psi(e) + \Psi(g)$, or there is $e_1 \leq e$, $e_1 \leq a \vee b$, $e_1 \neq 0$. In that case $e_1 \in D_{C_a}$ and as $e_1 \leq e \vee g$, we get $e_1 \leftrightarrow e \vee g$. Consequently, $e \vee g \leftrightarrow a$ and also $g \leftrightarrow a$, a contradiction. Suppose finally that $e \vee g \leftrightarrow a$, while $e \not\leftrightarrow a$, $g \not\leftrightarrow a$. Then $a \leq e \vee g$. Indeed, if there is $a_1 \leq a$, $a_1 \neq 0$, $a_1 \perp e \vee g$, then $a_1 \perp e$. It follows that $a_1 \leftrightarrow e$, and, as $a_1 \in D_{C_a}$, also $a \leftrightarrow e$, inconsistently with the assumption. We have shown that Ψ is a state on L.

Now we have $\Psi(d) = 1 = \Psi(b')$. Since $d \not\leftrightarrow b'$ and also $x \not\leftrightarrow b'$ for every $x \leq d$, $x \neq 0$, this is a contradiction with the Jauch–Piron property. We have proved that card $D_C \leq 1$ provided $C \neq C(L)$.

Step 2. Let $\mathscr{D} = \{D_C \mid C \in \mathscr{C}(L),\ \text{card } D_C = 1\} = \{D_1, D_2, \ldots, D_q\}$. Denote by d_i the (only) element of D_i, $i = 1, 2, \ldots, q$, and $E = \{d_1, d_2, \ldots, d_q\}$. Let $\mathscr{B} = \{B_1, B_2, \ldots, B_n\}$ be the set of all blocks of L. For each $B_i \in \mathscr{B}$ we denote $b_i = \vee(B_i \cap E)$. If $b_k \leftrightarrow b_l$ for $k \neq l$, and $b_k \neq b_l$, then there is $a \in E$ such that $a \leq b_k$ and $a \perp b_l$ (if necessary, we interchange the roles of b_k and b_l). By the unitality of L, there exists $\Phi \in \mathcal{S}(L)$ with $\Phi(a) = 1$. For this state we have $\Phi(b_k) = 1 = \Phi(b_l')$, but the only element of L under both b_k and b_l' is 0—a contradiction with the Jauch–Piron property. It follows that $b_k = b_l$.

We want to prove that $b_1 = b_2 = \cdots = b_n$. It suffices to show that $b_i \leftrightarrow b_j$ for all $i, j \in \{1, 2, \ldots, n\}$. Suppose, in contradiction, that $b_k \not\leftrightarrow b_l$ for some k, l. Notice that this assumption implies that any lower bound $a \in L$ of $\{b_1', b_2', \ldots, b_n'\}$ is different from b_i' for each $i = 1, 2, \ldots, n$. Let A be the set of all lower bounds of $\{b_1', b_2', \ldots, b_n'\}$ in L fulfilling the following condition: let $a_1 \in A$ and let a_2 be a lower bound of $\{b_1', b_2', \ldots, b_n'\}$ such that $a_1 \leq a_2$; then $\{a_1, a_2\} \subset C$ for some $C \in \mathscr{C}(L)$. Due to Lemma 2.3, for each lower bound $a_1 \in L$ of $\{b_1', b_2', \ldots, b_n'\}$ there is $a_2 \in A$, $a_2 \geq a_1$. For each $C \in \mathscr{C}(L)$ we denote $A_C = A \cap C$. We utilize again the compactness of $\mathcal{S}(L)$ to prove that, for each $C \in \mathscr{C}(L)$, there is $\Psi \in \mathcal{S}(L)$ such that $\Psi(b_i') = 1$, $i = 1, 2, \ldots, n$, and $\Psi(c) = 0$ for every $c \in D_{C(L)} \cup A_C$.

Realize, first, that $a' \wedge b_k' \not\leftrightarrow a' \wedge b_l'$ for each $a \in A$. For each $c \in D_{C(L)}$ and $a \in A_C[C \in \mathscr{C}(L)]$ we denote $\mathcal{S}_{c,a}(L) = \{\Phi \in \mathcal{S}(L) \mid \Phi(c) = 0, \Phi(a) = 0$, and $\Phi(b_i') = 1, i = 1, 2, \ldots, n\}$. If $c \in D_{C(L)}$ and $a \in A_C$, then there is $c_1 \in D_{C(L)}$, $c_1 \neq 0$, $c_1 \perp c \vee a$, and there exists $\Phi \in \mathcal{S}(L)$ with $\Phi(c_1) = 1$. We have $\Phi \in \mathcal{S}_{c,a}(L)$, and thus $\mathcal{S}_{c,a}(L)$ is a nonvoid closed subset of $\mathcal{S}(L)$. If $c_1, c_2 \in D_{C(L)}$ and $a_1, a_2 \in A_C$, then $c_1 \vee c_2 \in D_{C(L)}$, $a_1 \vee a_2 \in A_C$, and $\mathcal{S}_{c_1 \vee c_2, a_1 \vee a_2}(L) \subseteq \mathcal{S}_{c_1,a_1}(L) \cap \mathcal{S}_{c_2,a_2}(L)$. Hence $\bigcap_{(c,a) \in D_{C(L)} \times A_C} \mathcal{S}_{c,a}(L) \neq \emptyset$. This proves the existence of $\Psi \in \mathcal{S}(L)$ with $\Psi(b_i') = 1$ for $i = 1, 2, \ldots, n$, and $\Psi(c) = 0$ for every $c \in D_{C(L)} \cup A_C$.

We construct such a state for each $C \in \mathscr{C}(L)$ with $A_C \neq \emptyset$, denote them $\Psi_1, \Psi_2, \ldots, \Psi_p$, and put $\Psi = (1/p) \sum_{i=1}^{p} \Psi_p$. We have $\Psi(b_j') = 1$ for $j = 1, 2, \ldots, n$. If $a_1 \in L$ is a lower bound of $\{b_1', b_2', \ldots, b_n'\}$, then there is $C \in \mathscr{C}(L)$ and $a_2 \in A_C$ such that $a_2 \geq a_1$. If $\Phi \in \{\Phi_1, \Phi_2, \ldots, \Phi_p\}$ is the state corresponding to C, then $\Phi(a_1) = \Phi(a_2) = 0$ and thus $\Psi(a_1) \leq (p-1)/p < 1$. On the other hand, the Jauch–Piron property implies that there is a $a \in L$ such that $a \leq b_j'$ for each $j = 1, 2, \ldots, n$, and $\Psi(a) = 1$—a contradiction.

We proved that $b_i \leftrightarrow b_j$ for each $i, j \in \{1, 2, \ldots, n\}$ and therefore $b_1 = b_2 = \cdots = b_n = b$. Hence $b \in C(L)$ and we can write $L = [0, b] \times [0, b']$. It follows from Theorem 3.4 that $[0, b]$ and $[0, b']$ are both Jauch–Piron unital logics. But $[0, b]$ being finite, it is Boolean by Theorem 3.5. As there are no contral atoms in $[0, b]$, we have $[0, b] = \{0\}$ and $L = [0, b']$. ■

Theorem 3.7. Let L be a block-finite unital Jauch–Piron logic. Then L is a Boolean algebra.

Proof. Let us suppose that L is not Boolean. Define a relation $\sim$ on L as follows: $a \sim b$ if and only if there are $c \in L$ and $d, e \in D_{C(L)}$ such that

$a = c \vee d$, $b = c \vee e$. The relation $\sim$ is obviously symmetric and reflexive. We show that it is also transitive, and thus $\sim$ is an equivalence on L. Suppose that $a \sim b$ and $b \sim c$. There are $d, e \in L$ and $a_1, b_1, b_2, c_2 \in D_{C(L)}$ such that $a = d \vee a_1$, $b = d \vee b_1 = e \vee b_2$, $c = e \vee c_2$. Put $f = (d \wedge b_2) \vee a_1$, $g = (e \wedge b_1) \vee c_2$. We have $f, g \in D_{C(L)}$ (Lemma 2.2) and $a = (d \wedge e) \vee f$, $c = (d \wedge e) \vee g$, hence $a \sim c$.

We denote by P the quotient set $L/\sim$. We endow P with the orthocomplementation and the ordering inherited from L, i.e., if $[a], [b] \in P$, then $[a] = [b]'([a] \leq [b]$, resp.) if there are $a_1, b_1 \in L$, $a_1 \in [a]$, $b_1 \in [b]$ with $a_1 = b_1'$ ($a_1 \leq b_1$, resp.). It is a routine procedure to check that P is a logic. We denote by f the canonical logic morphism from L onto P.

We shall prove that f preserves the relation of compatibility, i.e., if $a, b \in L$, then $a \leftrightarrow b$ (in L) if and only if $f(a) \leftrightarrow f(b)$ (in P). Suppose that $a \leftrightarrow b$. We can write $a = c \vee e$, $b = d \vee e$ for $c, d, e \in L$, $c \perp d$, $c \perp e$, $d \perp e$. We have $f(c) \perp f(d)$, $f(c) \perp f(e)$, $f(d) \perp f(e)$, and $f(a) = f(c) \vee f(e)$, $f(b) = f(d) \vee f(e)$, i.e., $f(a) \leftrightarrow f(b)$. To prove the reverse implication, suppose $f(a) \leftrightarrow f(b)$. There are mutually orthogonal elements $f(c), f(d), f(e) \in P$ such that $f(a) = f(c) \vee f(e)$, $f(b) = f(d) \vee f(e)$. Further, there are mutually orthogonal elements $c_1, d_1, e_1 \in L$ such that $f(c_1) = f(c)$, $f(d_1) = f(d)$, and $f(e_1) = f(e)$. We have $c_1 \vee e_1 \leftrightarrow d_1 \vee e_1$, and, moreover,

$$(a \wedge (c_1 \vee e_1)') \vee (a' \wedge (c_1 \vee e_1)) \vee (b \wedge (d_1 \vee e_1)') \vee (b' \wedge (d_1 \vee e_1)) \in D_{C(L)}$$

Hence $a \leftrightarrow b$.

This result ensures that the system $\mathscr{E}(P)$ is isomorphic to the system $\mathscr{E}(L)$. More exactly, $f(a), f(b)$ are both elements of some $C_P \in \mathscr{E}(P)$ if and only if a, b are both elements of some $C_L \in \mathscr{E}(L)$. Moreover, if $a \in D_{C(L)}$, then $f(a) = 0$. Thus, there are no central bottom elements in P.

Now we shall prove that P is unital. Let $f(a) \in P, f(a) \neq 0$. Denote $L_a = \{x \in L \mid f(a) \leq f(x)\}$. Let $c, d \in L_a$. We show that there is $z \in L_a$, $z \leq c$, $z \leq d$. Suppose first that $f(c) = f(a)$. Then $f(c) \leq f(d)$, hence $c \leftrightarrow d$ and there are mutually orthogonal elements $c_1, d_1, e \in L$ such that $c = c_1 \vee e$, $d = d_1 \vee e$. Now also $f(c_1) \perp f(d_1)$, $f(c_1) \perp f(e)$, $f(d_1) \perp f(e)$, and $f(c) = f(c_1) \vee f(e)$, $f(d) = f(d_1) \vee f(e)$. Since $f(c) \leq f(d)$, we get $f(c_1) = 0$. Thus, $f(c \wedge d) = f(c) = f(a)$ and $c \wedge d \in L_a$. Let now $c, d \in L_a$ be arbitrary. We have $c \wedge a \in L_a$ and $f(c \wedge a) = f(a)$; hence also $z = (c \wedge a) \wedge d \in L_a$.

For every $c \in L_a$ denote $\mathscr{S}_c(L) = \{\Phi \in \mathscr{S}(L) \mid \Phi(c) = 1\}$. Obviously, for every $c \in L_a$, $\mathscr{S}_c(L)$ is a nonempty closed subset of $\mathscr{S}(L)$. Further, if $c_1, c_2 \in L_a$, then there is $c \in L_a$, $c \leq c_1$, $c \leq c_2$, and $\mathscr{S}_c(L) \subseteq \mathscr{S}_{c_1}(L) \cap \mathscr{S}_{c_2}(L)$. Due to the compactness of $\mathscr{S}(L)$, there is $\Phi \in \bigcap_{c \in L_a} \mathscr{S}_c(L)$. Now $\Phi(c) = 1$ for every $c \in L_a$, particularly $\Phi(a) = 1$ and $\Phi(x) = 1$ for every $x \in L$ such that $f(x) = 1$. According to Proposition 1.7, there exists a state $\Psi \in \mathscr{S}(P)$ with $\Psi(f(a)) = 1$.

We have shown that P is unital and block-finite. According to Theorem 3.3, P is a Jauch–Piron logic. Hence, by Lemma 3.6, all bottom elements in P should be central. But we have shown that there are no nonzero central bottom elements in P—a contradiction. The proof is finished. ■

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